# SPATIAL KINEMATICS OF GEARS IN ABSOLUTE COORDINATES

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## KEYWORDS

Dynamics, Multibody, Gears, Jacobian, Absolute Coordinates, Concave-convex, Rack, Pinion

## ABSTRACT

In this article the description of some types of spatial gear constraints (spur gears, bevel gears, etc.) in absolute coordinates is considered. Instead of the numerical expensive calculation of constraint Jacobian matrix we propose to use the transformed Jacobian matrix, which can be calculated much more efficiently. The proposed methods of description of gear constraints were implemented for the simulation of dynamics of CAD model of KUKA KR 15/2 industrial manipulator.

## INTRODUCTION

Gears and gearing systems are fundamental mechanical components, widely used in the design of machines and mechanical systems for the transmission of motion and forces. On the other hand the detailed description of gear constraints is usually out of interest of multibody literature. In a few books can be found the description of planar kinematics of gears (Haug 1989, Shabana 2001) or the description of the spatial gear kinematics in joint coordinates (Schweiger and Otter, 2003).

In this article the description of some types of spatial gear constraints (spur gears, bevel gears, etc.) in absolute coordinates is considered. The use of absolute coordinates helps to integrate dynamic simulation tools with CAD systems, widely used for design of mechanical systems.

In order to better understand the issues involved, it is useful to consider the equations of motion in absolute coordinates. Let $\mathbf{q}^i = (\mathbf{x}^{iT}, \mathbf{e}^{iT})^T$ be the vector of absolute coordinates of the $i$-th body consisting of position coordinates $\mathbf{x}^i = \left(x_1^i, x_2^i, x_3^i\right)^T$ and of orientation coordinates $\mathbf{e}^i$. Orientation coordinates can be defined in different ways (e.g. Euler angles, Bryan angles, Rodriguez parameters, Euler parameters, etc.). The vector of generalized velocities $\mathbf{v}^i = (\dot{\mathbf{x}}^{iT}, \boldsymbol{\omega}^{iT})^T$ includes linear velocity $\dot{\mathbf{x}}^i$ and angular velocity $\boldsymbol{\omega}^i$. The first derivative of $\mathbf{q}^i$ is proportional to the vector $\mathbf{v}^i$: $\dot{\mathbf{q}}^i$=$\mathbf{T}^i(\mathbf{q}^i)\mathbf{v}^i$, where $\mathbf{T}^i$ denote the relation matrix.

Let $\mathbf{q} = (\mathbf{q}^{1T} \;\; ... \;\; \mathbf{q}^{nT})^T$ be the vector of absolute coordinates of a multibody system. By $\mathbf{g}(\mathbf{q})$ denote the vector of constraints, describing joints, connecting bodies in the simulated mechanical system. By $\mathbf{G}(\mathbf{q})$ denote the Jacobian matrix of $\mathbf{g}(\mathbf{q})$:

$$\mathbf{G}(\mathbf{q}) = \frac{\partial \mathbf{g}(\mathbf{q})}{\partial \mathbf{q}} \tag{1}$$

Differentiating $\mathbf{g}(\mathbf{q})$, we get the equations of joint constraints on the velocity level:

$$\mathbf{g}_{\mathbf{v}}(\mathbf{q}, \mathbf{v}) = \mathbf{G}(\mathbf{q}) \cdot \dot{\mathbf{q}} = \mathbf{G}(\mathbf{q}) \cdot \mathbf{T}(\mathbf{q}) \cdot \mathbf{v} \tag{2}$$

Let $\widehat{\mathbf{G}}(\mathbf{q})=\mathbf{G}(\mathbf{q})\cdot\mathbf{T}(\mathbf{q})$ be the transformed Jacobian matrix. Then (2) can be written as

$$\mathbf{g}_{\mathbf{v}}(\mathbf{q}, \mathbf{v}) = \widehat{\mathbf{G}}(\mathbf{q}) \cdot \mathbf{v} = \mathbf{0} \tag{3}$$

We proposed (Vlasenko and Kasper 2009) the method of the simulation of multibodies, based on the Newton-Euler equations of motion

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{v}} + \widehat{\mathbf{G}}^T(\mathbf{q})\boldsymbol{\lambda} = \mathbf{f}(\mathbf{q}, \mathbf{v}) \tag{4}$$

where $\mathbf{f}(\mathbf{q}, \mathbf{v})$ is the vector of external forces, $\mathbf{M}$ is the mass matrix, $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers. The main advantage of this method is that it uses the matrix $\widehat{\mathbf{G}}(\mathbf{q})$, which can be calculated much easily than the original Jacobian matrix $\mathbf{G}(\mathbf{q})$. Furthermore, if the non-minimal set of coordinates is used (e.g. Euler parameters), then the size of $\widehat{\mathbf{G}}$ is less than the size of $\mathbf{G}$, that is important for the reduction of the simulation numerical costs.

In this article is shown the generation of constraints equations $\mathbf{g}$ and of transformed Jacobian matrix $\widehat{\mathbf{G}}$ for some types of gear joints, commonly used in mechanical systems (spur gears, bevel gears, etc.). In the description of gears we assume that constraints, generated by gear joints, limit only the relative rotation of gears. All other limitations on the relative motion of connected gears (e.g. constant distance between axes in the spur gear joint, etc.) are achieved as the result of connection of gears by other joints (usually by revolute joint) to some basement. This art of definition of gear constraints looks natural and similar to the definition of gear constraints in CAD-like systems (Autodesk Inventor, etc.).

The proposed methods of description of gear constraints were implemented for the simulation of dynamics of CAD model of KUKA KR 15/2 industrial manipulator.

## SPATIAL KINEMATICS OF GEARS

### Spur gear joint

*Equation of constraint on the coordinate level*

Let us consider the gear *i* and the gear *j*, shown in Figure 1, which roll relative to each other about parallel axes $\bar{\mathbf{a}}^i$ and $\bar{\mathbf{a}}^j$**.** Let $C^i$ and $C^j$ be the points on axes $\bar{\mathbf{a}}^i$ and $\bar{\mathbf{a}}^j$**,** lying on the line, perpendicular to $\bar{\mathbf{a}}^i$. We assume that the motion of gears is constrained in such way that only the relative rotation of them is allowed, i.e. $\bar{\mathbf{a}}^i$ remain parallel to $\bar{\mathbf{a}}^j$**,** the vector $C^iC^j$ remains perpendicular to $\bar{\mathbf{a}}^i$ and the distance between $C^i$ and $C^j$ remain constant (equal to the sum of the gears' radii).

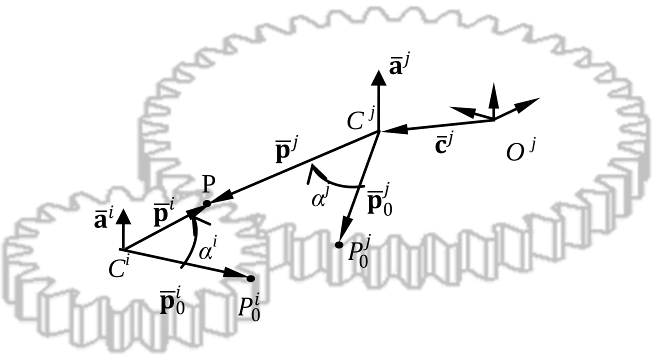


Figure 1: Spur Gear Joint

By $\bar{\mathbf{c}}^i$ and $\bar{\mathbf{c}}^j$ denote the local position vectors of $C^i$ and $C^j$**,** respectively. Let $P^i$ and $P^j$ be the points of contact on bodies *i* and *j*, lying on the line $C^iC^j$. By $P_0^j$ and $P_0^i$ denote the points of contact on gears at initial stage.

Gears roll relative to each other without slip, therefore, the arc length $P_0^iP^i$ and $P_0^jP^j$ of contact on the gears must be equal. Then we get the equation of constraint

$$\mathrm{g} = \alpha^j r^j + \alpha^i r^i \tag{5}$$

where $\alpha^i$ is the angle between $P_0^i$ and $P^i$**,** $\alpha^j$ is the angle between $P_0^j$ and $P^j$**,** $r^i$ and $r^j$ are the radii of gears.

Now we need to find the formula for the calculation of $\alpha^j$ and $\alpha^i$. Let $\bar{\mathbf{p}}_0^i, \bar{\mathbf{p}}^i$ be the vectors $C^iP_0^i$ and $C^iP^i$ expressed in body *i*. Let $\bar{\mathbf{p}}_0^j, \bar{\mathbf{p}}^j$ be the vectors $C^jP_0^j$ and $C^jP^j$ expressed in body *j*. From the coincidence of points $P^i$ and $P^j$ follows that

$$\mathbf{x}^i + \mathbf{R}^i\bar{\mathbf{c}}^i + \mathbf{R}^i\bar{\mathbf{p}}^i = \mathbf{x}^j + \mathbf{R}^j\bar{\mathbf{c}}^j + \mathbf{R}^j\bar{\mathbf{p}}^j \tag{6}$$

where $\mathbf{x}^i$ and $\mathbf{x}^j$ are the vector of position coordinates of bodies *i* and *j*, respectively; $\mathbf{R}^i$ and $\mathbf{R}^j$ are the transformation matrices of the two bodies. This can be also written in another form

$$\mathbf{x}^i + \mathbf{c}^i + \mathbf{p}^i = \mathbf{x}^j + \mathbf{c}^j + \mathbf{p}^j \tag{7}$$

where $\mathbf{c}^i = \mathbf{R}^i\bar{\mathbf{c}}^i$, $\mathbf{c}^j = \mathbf{R}^j\bar{\mathbf{c}}^j$, $\mathbf{p}^i = \mathbf{R}^i\bar{\mathbf{p}}^i$, $\mathbf{p}^j = \mathbf{R}^j\bar{\mathbf{p}}^j$ are the vectors $\bar{\mathbf{c}}^i, \bar{\mathbf{c}}^j, \bar{\mathbf{p}}^i, \bar{\mathbf{p}}^j$, expressed in the global frame.

Let $\mathbf{l}$ denote the vector of the constant length from $C^i$ to $C^j$, calculated as

$$\mathbf{l} = \mathbf{x}^j + \mathbf{c}^j - \mathbf{x}^i - \mathbf{c}^i \tag{8}$$

Then (7) can be rewritten as

$$\mathbf{p}^i - \mathbf{p}^j = \mathbf{l} \tag{9}$$

Let $\boldsymbol{e}^i$, $\boldsymbol{e}^j$ be the units vectors along $\mathbf{p}^i$ and $\mathbf{p}^j$, correspondently

$$\mathbf{p}^l = r^l\mathbf{e}^l \quad l = i,j \tag{10}$$

From the definition of vectors $\mathbf{p}^i$ and $\mathbf{p}^j$ follows that $\mathbf{e}^i$ and $\mathbf{e}^j$ can be calculated as

$$\mathbf{e}^i = -\mathbf{e}^j = \frac{\mathbf{l}}{\|\mathbf{l}\|} \tag{11}$$

In practice the joint is usually defined by the gear ratio

$$k = r^j/r^i \tag{12}$$

Then $r^i$, $r^j$ can be calculated from *k* using the formula

$$r^i = \|\mathbf{p}_0^i\| = \frac{1}{1+k}\cdot\|\mathbf{l}\| \tag{13}$$

$$r^j = \|\mathbf{p}_0^j\| = \frac{k}{1+k}\cdot\|\mathbf{l}\| \tag{14}$$

If the modulus of $\alpha^j$ and of $\alpha^i$ are less than $\pi/2$, then $\alpha^j$ and $\alpha^i$ can be calculated as

$$\alpha^l = \mathrm{asin}\left[\mathbf{a}^{lT}\cdot\left(\mathbf{e}_0^l \times \mathbf{e}^l\right)\right] \quad l = i,j \tag{15}$$

where $\mathbf{a}^l = \mathbf{R}^l\bar{\mathbf{a}}^l$ is the axis $\bar{\mathbf{a}}^l$**,** expressed in the global frame.

In practice we can guarantee that $\alpha^j$ and $\alpha^i$ are less than $\pi/2$ if during the simulation we always move the points $P_0^j$ and $P_0^i$ to the tops of last contacted teeth. The number of teeth of gears can be defined manually during the gear design or automatically by the simulation pre-compiler.

*The equation of constraint on the velocity level*

Let us show how we can find the equation of constraint on the velocity level $\mathbf{g_v}$ and to derive from $\mathbf{g_v}$ the formula for $\widehat{\mathbf{G}}$.

The equation of constraint on the velocity level are calculated as the time derivative of (5)

$$\mathrm{g}_{\mathbf{v}} = \dot{\alpha}^j r^j + \dot{\alpha}^i r^i = 0 \tag{16}$$

Let us define the relative orthogonal system of coordinates $C^i\mathbf{e}^{bx}\mathbf{e}^{by}\mathbf{e}^{bz}$ where the origin of the relative system is rigidly connected to $C^i$**,** $\mathbf{x}^b$ is the axes lying on the line $C^iC^j$**,** $\mathbf{e}^{by}$ lies on the axes $\bar{\mathbf{a}}^i$ (i.e. $\mathbf{e}^{by} = \bar{\mathbf{a}}^i$) and the vector $\mathbf{e}^{bz}$ is chosen in such way that the $C^i\mathbf{e}^{bx}\mathbf{e}^{by}\mathbf{e}^{bz}$ will be right-handed. Let $\mathbf{u}_b^{Pi}$**,** $\mathbf{u}_b^{Pj}$ be the velocities of points $P^i$ and $P^j$ relative $C^i\mathbf{e}^{bx}\mathbf{e}^{by}\mathbf{e}^{bz}$, expressed in the global coordinate system, respectively. From the definition of $C^i\mathbf{e}^{bx}\mathbf{e}^{by}\mathbf{e}^{bz}$ follows that

$$\boldsymbol{\omega}_b^l = \dot{\alpha}^l\mathbf{a}^l \quad l{=}i{,}j \tag{17}$$

where $\boldsymbol{\omega}_b^l$ be the relative angular velocity of $l$-th body, expressed in the global coordinate system, calculated as

$$\boldsymbol{\omega}_b^l = \boldsymbol{\omega}^l - \boldsymbol{\omega}^b \quad l=i,j \tag{18}$$

where $\boldsymbol{\omega}^b$ is the angular velocity of the frame $C^i\mathbf{e}^{bx}\mathbf{e}^{by}\mathbf{e}^{bz}$ and $\boldsymbol{\omega}^l$ is the angular velocity of the $l$-th body.

Cross-multiplying both parts of (17) by the vector $\mathbf{e}^l$ , we get

$$\boldsymbol{\omega}_b^l \times \mathbf{e}^l = \dot{\alpha}^l \mathbf{a}^l \times \mathbf{e}^l \quad l=i,j \tag{19}$$

Let $\mathbf{s}$ be a vector perpendicular to the axis of rotation $\mathbf{a}^i$ and to $\mathbf{p}^i$ : $\mathbf{s} = \mathbf{a}^i \times \mathbf{e}^i$ **.** Then from (19) follows that

$$\begin{aligned} \dot{\alpha}^i &= \mathbf{s}^T\left(\boldsymbol{\omega}_b^i \times \mathbf{e}^i\right) \\ \dot{\alpha}^j &= -\mathbf{s}^T\left(\boldsymbol{\omega}_b^j \times \mathbf{e}^j\right) \end{aligned} \tag{20}$$

Substituting $\dot{\alpha}^i$, $\dot{\alpha}^j$ in (16), we get

$$\mathrm{g}_{\mathbf{v}} = -\mathbf{s}^T\left(\boldsymbol{\omega}_b^j \times \mathbf{p}^j\right) + \mathbf{s}^T\left(\boldsymbol{\omega}_b^i \times \mathbf{p}^j\right) \tag{21}$$

Using the formula (18) for $\boldsymbol{\omega}_b^j, \boldsymbol{\omega}_b^i$, we obtain

$$\mathrm{g}_{\mathbf{v}} = \mathbf{s}^T\left(\boldsymbol{\omega}^i \times \mathbf{p}^i - \boldsymbol{\omega}^j \times \mathbf{p}^j + \boldsymbol{\omega}^b \times (\mathbf{p}^j - \mathbf{p}^i)\right) \tag{22}$$

Let $\mathbf{u}^{Ci}, \mathbf{u}^{Cj}$ be the absolute velocities of points $C^i$ and $C^j$ , respectively, calculated as

$$\mathbf{u}^{Cl} = \dot{\mathbf{x}}^l + \boldsymbol{\omega}^l \times \mathbf{c}^l \quad l=i,j \tag{23}$$

From the definition of $C^i\mathbf{e}^{bx}\mathbf{e}^{by}\mathbf{e}^{bz}$ follows the relation

$$\boldsymbol{\omega}^b \times (\mathbf{p}^i - \mathbf{p}^j) = \mathbf{u}^{Cj} - \mathbf{u}^{Ci} \tag{24}$$

Substituting this equation in (22) and using (23), we get:

$$\begin{aligned} \mathrm{g}_{\mathbf{v}} = \mathbf{s}^T\big(&\left[\dot{\mathbf{x}}^i + \boldsymbol{\omega}^i \times (\mathbf{c}^i + \mathbf{p}^i)\right] \\ &- \left[\dot{\mathbf{x}}^j + \boldsymbol{\omega}^j \times (\mathbf{c}^j + \mathbf{p}^j)\right]\big) \end{aligned} \tag{25}$$

The physical meaning of this equation is the equality of projections of velocities of $P^i$ and $P^j$ on the axis $\boldsymbol{s}$. Using the triple product formula

$$\mathbf{a}^T(\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \times \mathbf{a})^T\mathbf{b} = (\mathbf{a} \times \mathbf{b})^T\mathbf{c} \tag{26}$$

we get from (25)

$$\begin{aligned} \mathbf{g}_{\mathbf{v}} = \mathbf{s}^T\dot{\mathbf{x}}^i - [\mathbf{s} \times (\mathbf{c}^i + \mathbf{p}^i)]^T\boldsymbol{\omega}^i - \mathbf{s}^T\dot{\mathbf{x}}^j \\ + [\mathbf{s} \times (\mathbf{c}^j + \mathbf{p}^j)]^T\boldsymbol{\omega}^j = \mathbf{0} \end{aligned} \tag{27}$$

Now, using (3), we get the matrix $\widehat{\mathbf{G}}(\mathbf{q}^i,\mathbf{q}^j)$

$$\widehat{\mathbf{G}}^T = \begin{pmatrix} \mathbf{s} \\ -\mathbf{s} \times (\mathbf{c}^i + \mathbf{p}^i) \\ -\mathbf{s} \\ \mathbf{s} \times (\mathbf{c}^j + \mathbf{p}^j) \end{pmatrix} \tag{28}$$

Clear, that the calculation of $\widehat{\mathbf{G}}$ from (28) is much easier than the calculation of Jacobian $\mathbf{G}(\mathbf{q}^i,\mathbf{q}^j)$ from (1).

**Bevel gear joint**

Let us consider the gear $i$ and the gear $j$, shown in Figure 2, which roll relative to each other with intersecting axes $\bar{\mathbf{a}}^i$ and $\bar{\mathbf{a}}^j$ in such way that the angles of rotation $\alpha^i$, $\alpha^j$ are related as

$$\alpha^i = -\alpha^j k \tag{29}$$

where $k$ is the gear ratio defined by the quotient between the number of gear teeth.

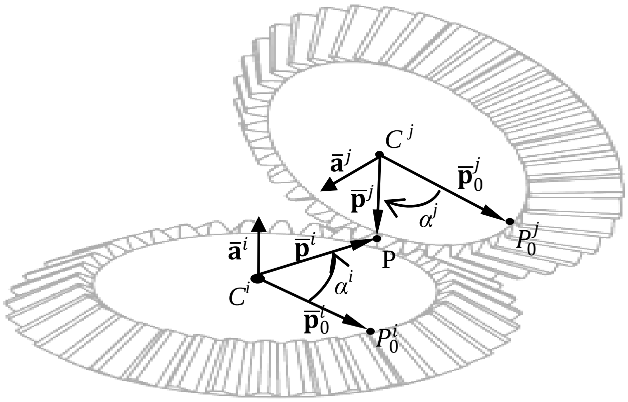


Figure 2: Bevel Gear Joint

In practice the joint is defined by the definition of bodies-fixed axes $\bar{\mathbf{a}}^i$ and $\bar{\mathbf{a}}^j$ and by the local position vectors $\bar{\mathbf{c}}^i$ and $\bar{\mathbf{c}}^j$ of some points $C^i$ and $\mathrm{C}^j$ on the axes $\bar{\mathbf{a}}^i$ and $\bar{\mathbf{a}}^j$, respectively.

By $\mathbf{l}$ denote the vector of the constant length from $C^i$ to $C^j$, calculated from (8). Let us define for each body $l$ some vector $\bar{\mathbf{p}}^l$ as perpendicular to $\bar{\mathbf{a}}^l$**,** which begins at $C^l$ and ends at the point $P^l$, lying on the line of contact. Let $r^i$ and $r^j$ be the modulus of $\bar{\mathbf{p}}^i$ and $\bar{\mathbf{p}}^j$ **,** respectively (similarly to the radii of $r^i$ and $r^j$ of spur gears).

Let us assume that $C^i$ and $\mathrm{C}^j$ are chosen in such way that points $P^i$ and $P^j$ are coincident, i.e.

$$\mathbf{p}^i - \mathbf{p}^j = \mathbf{l} \tag{30}$$

Then from the definition of $\bar{\mathbf{p}}^i$ and $\bar{\mathbf{p}}^j$ follows that

$$k = \frac{\|\mathbf{p}^j\|}{\|\mathbf{p}^i\|} = \frac{r^j}{r^i} \tag{31}$$

It can be shown that

$$\begin{aligned} r^i &= \frac{\left|\mathbf{a}^{jT}\mathbf{l}\right|}{\|\mathbf{a}^i \times \mathbf{a}^j\|} \\ r^j &= kr^i \end{aligned}$$

Now we can reformulate the equation of bevel gear constraint (29) in the form similar to the equation of the spur gear constraint

$$\mathrm{g} = \alpha^j r^j + \alpha^i r^i \tag{32}$$

Let us show now how to calculate the angles $\alpha^j$ and $\alpha^i$. By $P_0^j$ and $P_0^i$ denote the points $P^j$ and $P^i$ at initial stage and by $\bar{\mathbf{p}}_0^i, \bar{\mathbf{p}}_0^j$ the start values of vectors $\bar{\mathbf{p}}^i$ , $\bar{\mathbf{p}}^j$ , respectively. Obviously, the angle of relative rotation $\alpha^i$ is the angle between $\bar{\mathbf{p}}_0^i$ and $\bar{\mathbf{p}}^i$ and $\alpha^j$ is the angle between $\bar{\mathbf{p}}_0^j$ and $\bar{\mathbf{p}}^j$ .

From the numerical point of view it's easily to calculate $\alpha^l$ as the angle between unit vectors $\mathbf{e}^l = \mathbf{p}^l/r^l$, $\mathbf{e}_0^l = \mathbf{p}_0^l/r^l$ ($l$=$i$,$j$).

From the definition of vectors $\mathbf{p}^i$, $\mathbf{p}^i$ follows that $\mathbf{e}^i = \mathbf{s} \times \mathbf{a}^i$ and $\mathbf{e}^j = -\mathbf{s} \times \mathbf{a}^j$, where $\mathbf{s}$ is a unit vector, perpendicular to $\mathbf{a}^i, \mathbf{a}^j$, calculated as

$$\mathbf{s} = \frac{\mathbf{a}^i \times \mathbf{a}^j}{\|\mathbf{a}^i \times \mathbf{a}^j\|} \tag{33}$$

If the modulus of $\alpha^j$ and of $\alpha^i$ are less than $\pi/2$, then $\alpha^j$ and $\alpha^i$ can be calculated from (15).

Using the same procedure for the generation of equation of constraint on the velocity level, as it was used above for the spur gear case, we obtain that the transformed Jacobian matrix $\widehat{\mathbf{G}}$ can be calculated from (28).

**Concave-convex gear joint**

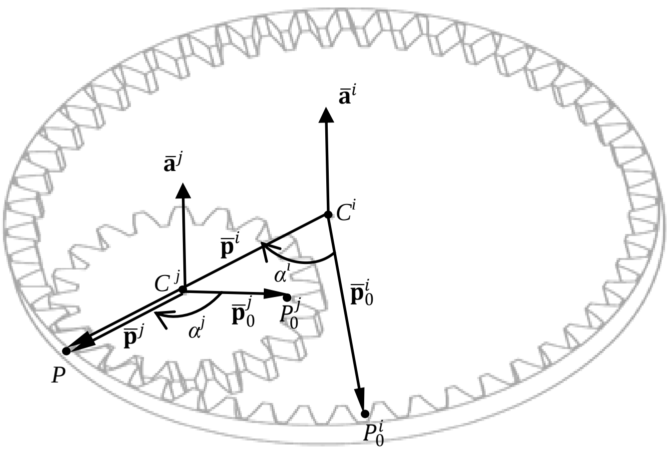


Figure 3: Concave-Convex Gear Joint

Let us consider a concave-convex gear joint, describing the rolling contact of smaller gear $j$ makes inside the larger interior gear $i$, shown Figure 3. Using the definitions from the spur gear case, we obtain the following relation between angles of rotations

$$\alpha^j r^j - \alpha^i r^i = 0 \tag{34}$$

Clear, that unlike convex gear case now the vectors $\mathbf{e}^i$ and $\mathbf{e}^j$ are equal and are calculated as

$$\mathbf{e}^i = \mathbf{e}^j = \frac{\mathbf{l}}{\|\mathbf{l}\|} \tag{35}$$

The formula for the calculation of radii $r^i$ and $r^j$ from the gear ratio $k = r^j / r^i$ also changes as

$$r^i = \|\mathbf{p}_0^i\| = \frac{1}{1-k} \cdot \|\mathbf{l}\|$$
$$r^j = \|\mathbf{p}_0^j\| = \frac{k}{1-k} \cdot \|\mathbf{l}\|$$

The angles $\alpha^j$ and $\alpha^i$ in (34) can be calculated from (15)

Using the same procedure, as it was used above for the spur gear case, we obtain the matrix $\widehat{\mathbf{G}}$

$$\widehat{\mathbf{G}}^T = \begin{pmatrix} -\mathbf{s} \\ \mathbf{s} \times (\mathbf{c}^i + \mathbf{p}^i) \\ \mathbf{s} \\ -\mathbf{s} \times (\mathbf{c}^j + \mathbf{p}^j) \end{pmatrix} \tag{36}$$

**Rack and pinion joint**

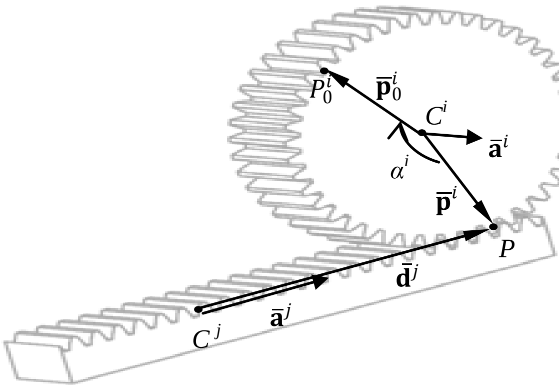


Figure 4: Rack and Pinion Joint

Let us consider a circular gear $i$ (called rack) and a flat bar $j$ (called pinion) constituting the constraint kinematic pair, shown in Figure 4. The rack rolls relative to the pinion about the axis $\bar{\mathbf{a}}^i$ whereby the pinion axis $\bar{\mathbf{a}}^j$ is situated on the line of contact. It's assumed that $\bar{\mathbf{a}}^i$ and $\bar{\mathbf{a}}^j$ are perpendicular in space, i.e. $\mathbf{a}^{iT}\mathbf{a}^j = \mathbf{0}$. Let $C^j$ be the point of contact on pinion at initial stage and $C^i$ be the points on axis $\bar{\mathbf{a}}^i$, chosen in such way that $C^iC^j$ is perpendicular to $\bar{\mathbf{a}}^i$. As before, we denote by $P^i$ and $P^j$ the points of contact on bodies $i$ and $j$, and by $P_0^i$ the point of contact on body $i$ at initial stage.

The equation of our constraint can be written as

$$\mathrm{g} = d^j - \alpha^i r^i \tag{37}$$

where $\alpha^i$ is the angle between $P_0^i$ and $P^i$, $d^j$ is the distance between $P_0^j$ and $P^j$. Let us show now how to calculate $d^j$ and $\alpha^i$.

Let $\bar{\mathbf{d}}^j$ be the vector from $P_0^j$ to $P^j$. It is obvious that

$$\mathbf{d}^j = d^j \mathbf{a}^j \tag{38}$$

Let $\bar{\mathbf{p}}_0^i, \bar{\mathbf{p}}^i, \bar{\mathbf{c}}^j$, be the local positions of $P_0^i$, $P^i$, $C^j$, respectively. From the coincidence of points $P^i$ and $P^j$ follows that

$$\mathbf{x}^i + \mathbf{c}^i + \mathbf{p}^i = \mathbf{x}^j + \mathbf{c}^j + \mathbf{d}^j \tag{39}$$

From the definition follows that $\mathbf{p}^i$ is perpendicular to $\mathbf{a}^j$. Therefore, multiplying (39) by $\mathbf{a}^j$, we get the formula for the calculation of $d^j$

$$d^j = \mathbf{a}^{jT}(\mathbf{x}^i + \mathbf{c}^i - \mathbf{x}^j - \mathbf{c}^j) \tag{40}$$

Let $\mathbf{e}^i = \mathbf{p}^i / r^i$ be the unit vector along $\mathbf{p}^i$. Let $\bar{\mathbf{e}}^j$ be the vector of coordinates of $\mathbf{e}^i$ in the pinion $j$. Obviously, $\bar{\mathbf{e}}^j$ is constant. The angle $\alpha^i$ can be calculated from (15), whereby the vector $\mathbf{e}^i$ can be calculated as $\mathbf{e}^i = \mathbf{R}^j \bar{\mathbf{e}}^j$.

In the same way as in the spur gear case, we obtain $\widehat{\mathbf{G}}$

$$\widehat{\mathbf{G}}^T = \begin{pmatrix} \mathbf{a}^j \\ \mathbf{a}^j \times (\mathbf{p}^i - \mathbf{c}^i) \\ -\mathbf{a}^j \\ \mathbf{a}^j \times (\mathbf{x}^i + \mathbf{c}^i - \mathbf{x}^j - \mathbf{p}^i) \end{pmatrix} \tag{41}$$

where $\mathbf{p}^i$ can be calculated from the equation $\mathbf{p}^i = r^i \mathbf{e}^i$ .

## SIMULATION EXAMPLE: INDUSTRIAL MANIPULATOR KUKA KR 15/2

In the last years we developed a component-oriented simulation software **V**irtual **S**ystems **D**esigner (VSD), integrated with CAD-like tool Autodesk Inventor (Kasper et. al 2007, Vlasenko and Kasper 2007). The proposed methods of description of gear joint constraints were implemented in VSD. As a test example we used an Autodesk Inventor model of the industrial manipulator KUKA KR 15/2, shown in Figure 5. This is a six-axis robot with articulated kinematics for all continuous-path controlled tasks. The main areas of application of KR 15/2 are handling, assembly, machining, etc. (Specifications of Robots)

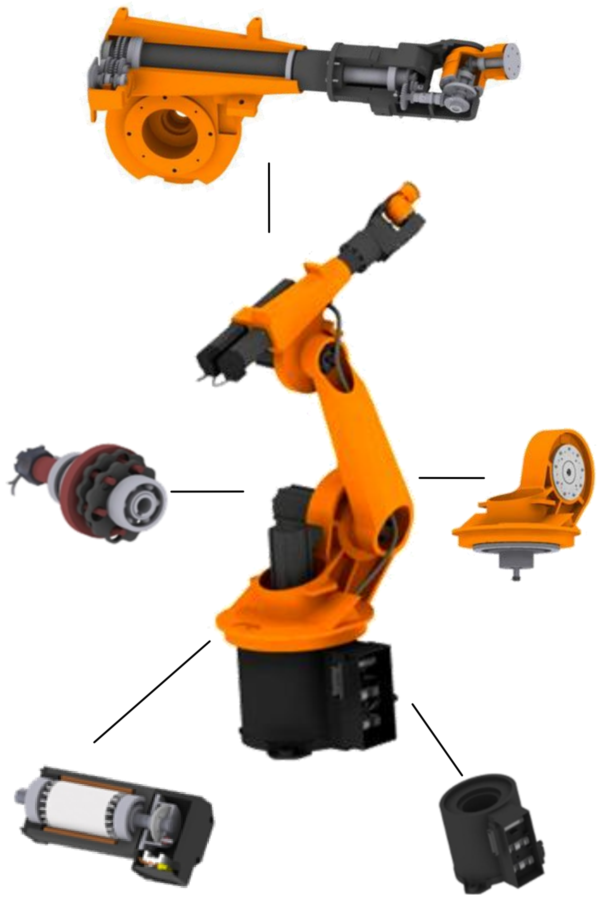

Figure 5: CAD Model of KUKA KR 15/2

The complete Autodesk Inventor model includes 1036 parts. The correspondent VSD model consists of 43 bodies connected by 95 joints (including 7 spur gear joints, 3 bevel gear joints and 10 concave-convex gear joints). Some of model constraints are redundant because of the model's design in Autesk Inventor (e.g. the definition of stiff connection as three plane-to-plane joints leads to the generation of three redundant constraints).

The dynamics of the manipulator under the action of gravitational force and of torques in motors is simulated. The numerical error of gear constraints on the coordinate and on the velocity levels are equall to the accuracy of used numerical methods. The analysis of monitored values of bodies' velocities and accelerations show the correctness of proposed methods for the generation of gear constraints.

## CONCLUSION AND FUTURE WORK

In this article is considered the spatial kinematics of most commonly used types of gear joints (spur gears, bevel gears, concave-convex gears and rack and pinion joints) in absolute coordinates. We show the methods of generation of equations of correspondent constraints on the coordinate and on the velocity level.

In standard methods of simulation of multibodies the calculation of constraint Jacobian matrix $\mathbf{G}$ is needed, which in the case of gears is a numerically expensive procedure. That is why we propose to use the simulation algorithm based on the calculation of transformed Jacobian matrix $\widehat{\mathbf{G}}$. In this article is shown that the generation of matrix $\widehat{\mathbf{G}}$ is easy in use and require a very small amount of computational effort. Moreover, the additional advantage $\widehat{\mathbf{G}}$ is its reduced size in comparison with $\mathbf{G}$ when the non-minimal set of orientation coordinates is used.

The proposed methods of description of gear constraints were implemented for the simulation of dynamics of KUKA KR 15/2 industrial manipulator. Test results show the correctness of proposed algorithms.

In future we plan to improve the area of implementation of proposed method by the description of other motion transmition elements (e.g. belt joints, cam-followers, etc.).

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